

# MULTIPLICITY BOUNDS FOR STEKLOV EIGENVALUES ON RIEMANNIAN SURFACES

MIKHAIL KARPUKHIN, GERASIM KOKAREV, AND IOSIF POLTEROVICH

**ABSTRACT.** We prove two explicit bounds for the multiplicities of Steklov eigenvalues  $\sigma_k$  on compact surfaces with boundary. One of the bounds depends only on the genus of a surface and the index  $k$  of an eigenvalue, while the other depends also on the number of boundary components. Using spectral asymptotics for pseudo-differential operators, we also show that on any given smooth Riemannian surface with boundary, the multiplicities of Steklov eigenvalues  $\sigma_k$  are uniformly bounded in  $k$ .

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Multiplicity bounds for Laplace eigenvalues.** Let  $M$  be a smooth closed surface. For a Riemannian metric  $g$  on  $M$  we denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leq \dots \lambda_k(g) \leq \dots$$

the eigenvalues of the Laplace operator  $-\Delta_g$ . A classical result by Cheng in [5] says that the multiplicities  $m_k(g)$  of these eigenvalues are bounded by quantities depending on the genus  $\gamma$  of  $M$  only. This result has been sharpened by Besson [4] for orientable surfaces, and by Nadirashvili [23] in the general case, to give the following version of the estimates for multiplicities:

$$(1.1.1) \quad m_k(g) \leq 2(2 - \chi) + 2k + 1, \quad k = 0, 1, \dots,$$

where  $\chi$  is the Euler-Poincaré number of  $M$ . When  $M$  is a surface of zero genus these bounds are sharp for the first eigenvalue.

The purpose of this paper is to prove analogues of such bounds for boundary value problems on surfaces. Such results seem to be missing in the literature, apart from the orientable zero genus case considered in [23, 18, 17] and [1], see also Remark 1.3.6. We are essentially concerned with the Steklov eigenvalue problem; problems with other boundary conditions are discussed at the end of the paper.

**1.2. Steklov eigenvalue problem.** From now on let  $(M, g)$  be a compact Riemannian surface with a non-empty boundary. For a given bounded non-negative function  $\rho$  on the boundary  $\partial M$  the Steklov eigenvalue problem is stated as:

$$(1.2.1) \quad \Delta u = 0 \quad \text{in } M, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \sigma \rho u \quad \text{on } \partial M,$$

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where  $\nu$  is an outward normal. Denote by  $\mu$  an absolutely continuous measure on the boundary  $\partial M$  with the density  $\rho$ , that is  $d\mu = \rho ds_g$ . The real numbers  $\sigma$  for which the solution above exists are eigenvalues of the Dirichlet form  $\int |\nabla u|^2 dVol_g$  in the space  $L_2(M, \mu)$ . Its spectrum is non-negative and discrete, see [2], and we denote by

$$0 = \sigma_0(g, \mu) < \sigma_1(g, \mu) \leq \dots \sigma_k(g, \mu) \leq \dots$$

the corresponding eigenvalues. This eigenvalue problem was considered in 1902 by Steklov and since then was studied extensively; we refer to [2] and the recent papers [14, 9] for a comprehensive list of references on the subject.

**1.3. Main results.** Our main result is the following theorem.

**Theorem 1.3.1.** *Let  $(M, g)$  be a compact Riemannian surface with a non-empty boundary. Let  $\mu$  be an absolutely continuous measure on  $\partial M$  whose density is bounded. Then the multiplicity  $m_k(g, \mu)$  of the Steklov eigenvalue  $\sigma_k(g, \mu)$  satisfies the inequalities*

$$(1.3.2) \quad m_k(g, \mu) \leq 2(2 - \bar{\chi}) + 2k + 1,$$

$$(1.3.3) \quad m_k(g, \mu) \leq 2(2 - \bar{\chi}) + 2l + k,$$

where  $\bar{\chi} = \chi + l$ , and  $\chi$  and  $l$  stand for the Euler-Poincare number and the number of the boundary components of  $M$  respectively.

Note that  $\bar{\chi}$  in the theorem depends on the genus  $\gamma$  of  $M$  only. More precisely, it equals  $2 - 2\gamma$  or  $2 - \gamma$  with the respect to the cases  $M$  is orientable or not. Both inequalities (1.3.2) and (1.3.3) are similar in a way to the Besson-Nadirashvili multiplicity bounds for Laplace eigenvalues on closed surfaces. The right hand-side of (1.3.2) is the same function of the genus of  $M$  as in (1.1.1). This bound does not depend on any boundary data and, as we show, holds for other boundary value problems, see section 5. The second equality can be re-written in the form

$$m_k(g, \mu) \leq 2(2 - \chi) + k;$$

it is specific to the Steklov problem, and for  $k > 2l - 1$  is sharper than the first one. The proofs of both inequalities are built on the ideas due to [23, 18] and use the properties of nodal graphs. In comparison with other classical boundary value problems there is an additional difficulty related to the fact that for a Steklov eigenfunction there is no known local model for the nodal set at the boundary points. However, we show that, unlike for general harmonic functions, the nodal graph of a Steklov eigenfunction is still finite.

For an annulus inequality (1.3.2) yields  $m_1(g, \mu) \leq 3$ ; this bound is sharp, and the equality is attained on a “critical catenoid” constructed in [9]. In general, Theorem 1.3.1 does not give sharp multiplicity bounds. It is an interesting question to understand whether  $m_k(g, \mu)$  is uniformly bounded in all parameters; see section 1.4. More specifically, one may ask the following question, cf. [15, Question 1.8]:

**Question 1.3.4.** *Does there exist a sequence of surfaces  $(M_n, g_n)$  with boundary measures  $\mu_n$  such that  $m_1(g_n, \mu_n) \rightarrow \infty$  as  $n \rightarrow +\infty$ ?*

If such a sequence exists, by inequality (1.3.2) the corresponding genera  $\gamma_n$  of  $M_n$  tend to infinity. Note also that the answer to an analogous question for the multiplicity of the first Laplace eigenvalue is positive [6].

*Remark 1.3.5.* For a disk, Alessandrini and Magnanini proved in [1] the bound  $m_k(g, \mu) \leq 2k$ , which is sharp for the first eigenvalue. In comparison, our result gives the bound  $m_k(g, \mu) \leq k + 2$ . Moreover, it can be slightly improved even further: for an even  $k$  we show that  $m_k(g, \mu) \leq k + 1$ , see Remark 3.3.2.

*Remark 1.3.6.* While the present paper was at the final stage of preparation, a different proof of inequality (1.3.2) for orientable surfaces appeared in [10, 20]. The approaches behind all the proofs go back to the ideas of Cheng and Besson. At the same time, our proof that the nodal graph is finite is different from the one in [10]: it is based on a topological argument and uses only Courant's nodal domain theorem. Besides, it applies to general boundary measures  $\mu$ , see Lemma 3.1.1 and the discussion in section 5. Note also that for non-orientable surfaces, inequality (1.3.2) is sharper than the estimate in [10, 20].

**1.4. Asymptotic bounds for Steklov eigenvalues.** Suppose that the boundary  $\partial M$  is smooth, and the weight function  $\rho$  in (1.2.1) is smooth and strictly positive. Then the Steklov eigenvalues can be viewed as the eigenvalues of a self-adjoint elliptic pseudo-differential operator of the first order; it sends a function on  $\partial M$  to the normal derivative of its harmonic extension multiplied by  $\rho^{-1}$ . In particular, for  $\rho \equiv 1$ , this pseudo-differential operator is precisely equal to the Dirichlet-to-Neumann operator on  $\partial\Omega$ . Using Hörmander's theorem on spectral asymptotics for pseudo-differential operators [19], we obtain the following result.

**Theorem 1.4.1.** *Let  $(M, g)$  be a compact Riemannian surface with a smooth boundary and  $\mu$  be a measure on  $\partial M$  whose density  $\rho$  is smooth and strictly positive. Then the multiplicities  $m_k(g, \mu)$  of Steklov eigenvalues are uniformly bounded in  $k$ , i.e. there exists a constant  $C_{g, \mu}$ , depending on a metric  $g$  and a measure  $\mu$ , such that*

$$m_k(g, \mu) \leq C_{g, \mu} \quad \text{for all } k = 0, 1, \dots$$

The version of this result for Laplace eigenvalues is well-known, see [17]. When  $M$  is a disk, Theorem 1.4.1 can be strengthened to the following statement.

**Proposition 1.4.2.** *Under the hypotheses of Theorem 1.4.1, suppose that  $M$  is homeomorphic to a disk. Then there exists an integer  $K_{g, \mu} > 0$ , depending on a metric  $g$  and a measure  $\mu$ , such that  $m_k(g, \mu) \leq 2$  for all  $k > K_{g, \mu}$ .*

Note that the inequality above is sharp; it is attained on the unit Euclidean disk. The proof of Proposition 1.4.2 uses the uniformisation theorem and the known asymptotics [24, 8] for the Steklov eigenvalues of the unit Euclidean disk.

## 2. PRELIMINARIES

**2.1. Variational principle and Courant's nodal domain theorem.** We start with recalling a variational setting for the Steklov eigenvalue problems. Given a Riemannian surface  $(M, g)$  and a measure  $\mu$  on its boundary, the Steklov eigenvalues

can be defined by the min-max principle

$$\sigma_k(g, \mu) = \inf_{\Lambda^{k+1}} \sup_{u \in \Lambda^{k+1}} R_g(u, \mu),$$

where the infimum is taken over all  $(k+1)$ -dimensional subspaces  $\Lambda^{k+1} \subset L_2(M, \mu)$  formed by  $C^\infty$ -smooth functions, the supremum is over non-trivial  $u \in \Lambda^{k+1}$ , and  $R_g(u, \mu)$  stands for the Rayleigh quotient

$$R_g(u, \mu) = \left( \int_M |\nabla u|^2 dVol_g \right) / \left( \int_M u^2 d\mu \right).$$

Mention that the natural space of test-functions for the Rayleigh quotient above is

$$(2.1.1) \quad \mathcal{L} = L_2(M, \mu) \cap L_2^1(M, Vol_g);$$

here the second space in the intersection is formed by distributions whose derivatives are in  $L_2(M, \mu)$ , see [22]. The Steklov eigenfunctions can be then regarded as solutions of the equation

$$(2.1.2) \quad \int_M \langle \nabla u, \nabla \varphi \rangle dVol_g = \sigma_k(g, \mu) \int_M u \varphi d\mu$$

understood as an integral identity, where the test-function  $\varphi$  ranges in  $\mathcal{L}$ . The equation above can be also viewed as a Schrodinger equation whose potential is a measure supported on the boundary of  $M$ .

Let  $u$  be a Steklov eigenfunction. It is harmonic inside  $M$ , and, in particular, is  $C^\infty$ -smooth. By  $\mathcal{N}(u)$  we denote its nodal set, that is the set  $u^{-1}(0)$ . Recall that a connected component of  $M \setminus \mathcal{N}(u)$  is called its nodal domain. By maximum principle, it is straightforward to conclude that the closure of each nodal domain has a non-trivial intersection with the boundary  $\partial M$ . Further, by the strong maximum principle [12] any Steklov eigenfunction has different signs on adjacent nodal domains. For the sequel we need a version of Courant's nodal domain theorem for the Steklov eigenfunctions; see [21] for the case of zero genus surfaces.

**Courant's nodal domain theorem.** Let  $(M, g)$  be a smooth compact Riemannian surface with boundary, and  $\mu$  be an absolutely continuous Radon measure on  $\partial M$  whose density is bounded. Then each Steklov eigenfunction  $u$  corresponding to the eigenvalue  $\sigma_k(g, \mu)$  has at most  $(k+1)$  nodal domains.

*Proof.* Suppose the contrary. Then there exists more than  $(k+1)$  nodal domains  $G_1, \dots, G_{k+1}, G_{k+2}, \dots$  of an eigenfunction  $u$ . Consider the functions

$$\varphi_i = \begin{cases} u|_{G_i}, & \text{on } G_i, \\ 0, & \text{on } M \setminus G_i. \end{cases}$$

Since  $u$  is continuous and vanishes on  $\partial G_i \setminus \partial M$ , it is straightforward to conclude that  $\varphi_i \in \mathcal{L}$ . Using it as a test-function in equation (2.1.2), we obtain that

$$(2.1.3) \quad \int_M |\nabla \varphi_i|^2 dVol_g = \lambda_k(\mu, g) \int_M \varphi_i^2 d\mu.$$

Now consider the sum

$$\psi = \sum_{i=1}^{k+1} \alpha_i \varphi_i.$$

Since the closure of all nodal domains intersect  $\partial M$ , after an appropriate choice of the  $\alpha_i$ 's one can suppose that  $\psi$  is orthogonal in  $L_2(M, \mu)$  to all lower eigenfunctions. Then by definition of the eigenvalue  $\lambda_k(\mu, g)$  we obtain

$$\lambda_k(\mu, g) \leq R_g(\psi, g) \leq \lambda_k(\mu, g),$$

where in the last inequality we used relation (2.1.3). Thus, we see that  $\psi$  has to be an eigenfunction itself. However, since it vanishes identically on  $G_{k+2}$ , it has to vanish identically everywhere by the unique continuation property [12]. This contradiction finishes the proof.  $\square$

**2.2. Local behaviour of harmonic functions; vanishing order.** Let  $u$  be a harmonic function on  $M$  and  $x \in M$  be an interior point. By the *vanishing order* of  $u$  at  $x$  we mean a non-negative integer, denoted by  $\text{ord}_x(u)$ , that is the order of the first non-vanishing derivative of  $u$  at  $x$ . The following statement is classical, see [3] and [16, Theorem 4.1], and holds for solutions of rather general second order linear elliptic equations.

**Proposition 2.2.1.** *Let  $(M, g)$  be a smooth compact Riemannian surface with boundary, and  $u$  be a harmonic function on  $M$ . Then for any interior point  $x_0 \in M$  there exists a non-trivial homogeneous harmonic polynomial  $P_n$  of degree  $n = \text{ord}_{x_0}(u)$ , defined in a neighbourhood  $U$  of  $x_0$ , such that*

$$u(x) = P_n(x - x_0) + O(|x - x_0|^{n+1}),$$

where  $x \in U$ .

In the proposition above we assume that the neighbourhood  $U$  is such that the metric  $g|_U$  is conformally Euclidean. In particular, the property of being harmonic on  $U$  with respect to the metric  $g$  is equivalent to being harmonic with respect to the Euclidean metric. Now for a given positive integer  $\ell$  consider the set

$$\mathcal{N}^\ell(u) = \{x \in M \mid \text{ord}_x(u) \geq \ell\}.$$

Using Prop. 2.2.1, in [5] Cheng shows that around a point  $x_0 \in \mathcal{N}(u)$  the nodal set is diffeomorphic to the nodal set of the corresponding harmonic polynomial  $P_n$ , which consists of  $n = \text{ord}_{x_0}(u)$  lines meeting at the origin. In particular, the set  $\mathcal{N}^2(u)$  consists of isolated points in the interior of  $M$ , and the complement  $\mathcal{N}^1(u) \setminus \mathcal{N}^2(u)$  is a collection of  $C^\infty$ -smooth arcs. Thus, the nodal set  $\mathcal{N}(u)$  can be viewed as a graph in the interior of  $M$  whose vertices are points  $x \in \mathcal{N}^2(u)$  and edges are connected components of  $\mathcal{N}^1(u) \setminus \mathcal{N}^2(u)$ . In sequel we refer to  $\mathcal{N}(u)$  as the *nodal graph*, meaning this graph structure. Mention that there are harmonic functions whose nodal graph is not finite.

The following statement highlights the difference between Steklov eigenfunctions and arbitrary harmonic functions; it is a consequence of Lemma 3.1.1 in the next section.

**Proposition 2.2.2.** *Let  $(M, g)$  be a smooth compact Riemannian surface with boundary, and  $\mu$  be an absolutely continuous Radon measure on  $\partial M$  whose density is bounded. Then the nodal graph  $\mathcal{N}(u)$  of a non-trivial Steklov eigenfunction  $u$  has a finite number of vertices and edges.*

Mention that Prop. 2.2.2 does not bar possible pathologies of nodal set behaviour that do not occur for other classical boundary value problems. For example, it does not say that the closure of  $\mathcal{N}(u)$  intersects  $\partial M$  by a finite set.

**2.3. Graphs in surfaces: basic background.** The purpose of this subsection is to introduce notation and collect a number of auxiliary facts used throughout the rest of the paper. Let  $S$  be a surface, possibly non-compact. Recall that a graph  $\Gamma \subset S$  is a collection of points, called *vertices*, and embedded intervals, called *edges*, such that the boundary of each edge belongs to the set of vertices. In addition, we assume that edges do not intersect and do not contain vertices. A graph is called *compact* if it is compact as a subset; it is called *finite*, if it has a finite number of vertices and edges. For example, for a non-trivial Steklov eigenfunction  $u$  the nodal graph  $\mathcal{N}(u)$ , viewed as a subset in the interior of  $M$ , is not compact, since it contains edges approaching the boundary.

Let  $\Gamma$  be a finite compact graph in  $S$ . For a vertex  $x \in \Gamma$  its *degree*  $\deg_\Gamma(x)$  is the number of edges incident to  $x$ ; if there is an edge that starts and ends at  $x$ , then it counts twice. The number of edges  $e$  of a finite compact graph is given by the formula

$$(2.3.1) \quad 2e = \sum \deg_\Gamma(x),$$

where the sum is taken over all vertices  $x \in \Gamma$ . Connected components of  $S \setminus \Gamma$  are called *faces* of  $\Gamma$ . The following inequality is a consequence of the standard Euler formula for a cell complex, see [11, p. 207].

**The Euler inequality.** Let  $\Gamma$  be a finite graph in a closed surface  $S$ , and  $v$ ,  $e$ , and  $f$  be the number of its vertices, edges, and faces respectively. Then the following inequality holds:

$$(2.3.2) \quad v - e + f \geq \chi,$$

where  $\chi$  is the Euler-Poincare number of  $S$ .

Finally, we recall the terminology for paths in graphs, which is used at the end of Sect. 3. By a path in a graph  $\Gamma$  we mean a continuous map  $\phi : [0, 1] \rightarrow \Gamma$  such that  $\phi(0)$  and  $\phi(1)$  are vertices, and if the image of  $\phi$  intersects non-trivially with an edge, then it contains this edge. A path in  $\Gamma$  is called *simple*, if it has no repeated vertices and edges. A closed path in a finite graph is called the *simple cycle*, or *circuit*, if it has no repeated vertices and edges except for  $\phi(0) = \phi(1)$ . A *tree* is a connected graph that has no circuits; its every two vertices can be joined by a simple path.

### 3. PROOF OF THEOREM 1.3.1

**3.1. Reduced nodal graph.** Let  $M$  be a smooth Riemannian surface with a non-empty boundary and  $\bar{M}$  be a closed surface of the same genus, viewed as the image of  $M$  under collapsing its boundary components to points. By  $\bar{\mathcal{N}}(u)$  we denote the corresponding image of a nodal graph  $\mathcal{N}(u)$ ; we call it the *reduced nodal graph*. More precisely, its edges are the same nodal arcs, and there are two types of vertices: vertices that correspond to the boundary components that contain limit

points of nodal lines, *vertices of the first kind*, and genuine vertices that correspond to the points in  $\mathcal{N}^2(u)$ , *vertices of the second kind*. It is straightforward to see that the number of nodal domains of an eigenfunction  $u$  is precisely the number of the connected components of  $\bar{M} \setminus \mathcal{N}(u)$ . Through out the paper we use the notation  $\bar{\chi}$  for the Euler-Poincare number of  $\bar{M}$ . It coincides with the quantity  $\chi + l$ , used in Theorem 1.3.1, and is called the *reduced Euler-Poincare number* of  $M$ .

The following lemma is a basis for the proof of Theorem 1.3.1. It uses only Courant's nodal domain theorem, and holds for eigenfunctions of rather general boundary value problems.

**Lemma 3.1.1.** *Let  $(M, g)$  be a smooth compact Riemannian surface with boundary, and  $\mu$  be an absolutely continuous Radon measure on  $\partial M$  whose density is bounded. Then the reduced nodal graph  $\bar{\mathcal{N}}(u)$  of a non-trivial Steklov eigenfunction  $u$  is finite, i.e. it has a finite number of vertices and edges.*

*Proof.* Consider the reduced nodal graph  $\bar{\mathcal{N}}(u)$  corresponding to a non-trivial Steklov eigenfunction  $u$ . To prove the lemma it is sufficient to rule out the occurrence of:

- (i) infinite degree vertices of the first kind and
- (ii) the infinite number of vertices of the second kind

in  $\bar{\mathcal{N}}(u)$ . We are going to construct new graphs in  $\bar{M}$  by resolving vertices of  $\bar{\mathcal{N}}(u)$  in the following fashion. Let  $x \in \mathcal{N}^2(u)$  be a vertex of the second kind; its degree equals  $2n$ , where  $n = \text{ord}_x(u)$ . Let  $U$  be a coordinate ball centered at  $x$  that does not contain other vertices and such that nodal arcs incident to  $x$  intersect  $\partial U$  at  $2n$  points precisely. We denote these points by  $y_i$ , where  $i = 0, \dots, 2n-1$ , and assume that they are ordered consequently in the clockwise fashion. A new graph is obtained from  $\bar{\mathcal{N}}(u)$  by changing it inside  $U$  and removing possibly appeared edges without vertices. More precisely, we remove the nodal set inside  $U$  and round-off the edges on the boundary  $\partial U$  by non-intersecting arcs in  $U$  joining points  $y_{2j}$  and  $y_{2j+1}$ . If there was an edge that starts and ends at  $x$ , then such a procedure may make it into a loop. If this occurs, then we remove this loop to obtain a genuine graph in  $\bar{M}$ . It has one vertex less and at most as many faces as the original graph.

*Ruling out (i).* Let us resolve each vertex of the second kind in  $\bar{\mathcal{N}}(u)$  in the way described above. The result is a graph  $\Gamma$  in  $\bar{M}$  whose only vertices are the first kind vertices in  $\bar{\mathcal{N}}(u)$ ; we denote by  $v$  their number. Besides, it has at most as many faces as the reduced nodal graph, that is by Courant's nodal domain theorem at most  $k+1$ . Suppose that the reduced nodal graph has an infinite degree vertex of the first kind; then so does  $\Gamma$ . Let us remove all edges in  $\Gamma$  except for  $v+k+2-\bar{\chi}$  of them to obtain a new finite graph, and denote by  $f$  the number of its faces. Since removing an edge does not increase the number of faces, we have  $f \leq k+1$ . On the other hand, by the Euler inequality (2.3.2), we have

$$f \geq e - v + \bar{\chi} = k + 2.$$

Thus, we arrive at a contradiction.

*Ruling out (ii).* Suppose the contrary; the situation described in (ii) occurs. Let  $v$  be a number of vertices of the first kind in  $\mathcal{N}(u)$ . Let us resolve all vertices of the second kind except for  $v+k+2-\bar{\chi}$  of them. The result is a finite graph  $\Gamma'$ . Denote by  $v'$ ,  $e'$ , and  $f'$  the number of its vertices, edges, and faces respectively; then we have

$$v' \leq 2v+k+2-\bar{\chi} \quad \text{and} \quad e' \geq 2(v+k+2-\bar{\chi}).$$

Here in the second inequality we used formula (2.3.1) and the fact that the degree of each vertex  $x \in \mathcal{N}^2(u)$  is at least 4. Combining the first two inequalities with the Euler inequality (2.3.2), we obtain

$$f' \geq e' - v' + \bar{\chi} \geq k+2.$$

Thus, we arrive at a contradiction with the Courant's nodal domain theorem.  $\square$

**3.2. Multiplicity bounds: the first inequality.** We start with a lemma that gives a lower bound for the number of nodal domains via the vanishing order of points  $x \in \mathcal{N}^2(u)$ . For the Dirichlet boundary problem on surfaces of zero genus it is proved in [18].

**Lemma 3.2.1.** *Let  $(M, g)$  be a smooth compact Riemannian surface with boundary, and  $\mu$  be an absolutely continuous Radon measure on  $\partial M$  whose density is bounded. Then for any non-trivial Steklov eigenfunction  $u$  the number of its nodal domains is at least  $\sum(\text{ord}_x(u) - 1) + \bar{\chi}$ , where the sum is taken over all points in  $\mathcal{N}^2(u)$  and  $\bar{\chi}$  is the reduced Euler-Poincare number of  $M$ .*

*Proof.* Let  $\tilde{\mathcal{N}}(u)$  be a reduced nodal graph in  $\tilde{M}$ , and  $v$ ,  $e$ , and  $f$  be the number of its vertices, edges, and faces respectively; by  $r$  we denote the number of first kind vertices. Using formula (2.3.1), we get

$$e \geq r + \sum \text{ord}_x(u),$$

where the sum is taken over  $x \in \mathcal{N}^2(u)$ . Here we used the fact that the degree of each vertex of the first kind is at least two. Viewing  $v$  as the sum  $r + \sum 1$ , where the sum symbol is again over  $x \in \mathcal{N}^2(u)$ , by the Euler inequality we obtain

$$f \geq e - v + \bar{\chi} \geq \sum(\text{ord}_x(u) - 1) + \bar{\chi}.$$

Since  $f$  is the precisely the number of nodal domains, we are done.  $\square$

The following lemma is a version of the statement due to [23].

**Lemma 3.2.2.** *Let  $u_1, \dots, u_{2n}$  be a collection of non-trivial linearly independent harmonic functions. Then for a given interior point  $x \in M$  there exists a non-trivial linear combination  $\sum \alpha_i u_i$  whose vanishing order at the point  $x$  is at least  $n$ .*

*Proof.* Let  $V$  be the span of  $u_1, \dots, u_{2n}$ , and  $V_i$  be its subspace formed by functions  $u \in V$  whose vanishing order at  $x$  is at least  $i$ ,  $\text{ord}_x(u) \geq i$ . Clearly, the subspaces  $V_i$  form a nested sequence,  $V_{i+1} \subset V_i$ . The statement of the lemma says that  $V_n$  is non-trivial. Suppose the contrary, that is  $V_n = \{0\}$ . Then the dimension of  $V$  is given by the formula

$$\dim V = 1 + \sum_{i=1}^{n-1} \dim(V_i/V_{i+1}).$$



By Proposition 2.2.1 the factor-space  $V_i/V_{i+1}$  can be identified with a subspace of homogeneous harmonic polynomials of order  $i$ . In polar coordinates on  $\mathbf{R}^2$  such polynomials have the form

$$P_i(r \cos \theta, r \sin \theta) = ar^i \cos(n\theta) + br^i \sin(n\theta),$$

and in particular, form a space of dimension two. Thus, we obtain

$$\dim V \leq 1 + 2(n - 1) = 2n - 1.$$

This is a contradiction with the hypotheses of the lemma.  $\square$

Now we prove the first inequality in Theorem 1.3.1:

$$m_k(g, \mu) \leq 2(2 - \bar{\chi}) + 2k + 1.$$

Suppose the contrary to its statement. Then there exists at least  $2(2 - \bar{\chi}) + 2k + 2$  linearly independent eigenfunctions corresponding to the eigenvalue  $\sigma_k(g, \mu)$ . Pick an interior point  $x \in M$ . By Lemma 3.2.2 there exists a new eigenfunction  $u$  whose vanishing order at the point  $x$  is at least  $2 - \bar{\chi} + k + 1$ . Combining this with Lemma 3.2.1, we conclude that the number of the nodal domains of  $u$  is at least  $k + 2$ . Thus, we arrive at a contradiction with Courant's nodal domains theorem.  $\square$

**3.3. Multiplicity bounds: the second inequality.** The proof of the second inequality is based on the following version of Lemma 3.2.1.

**Lemma 3.3.1.** *Let  $(M, g)$  be a smooth compact Riemannian surface with boundary, and  $\mu$  be an absolutely continuous Radon measure on  $\partial M$  whose density is bounded. Then for any non-trivial Steklov eigenfunction  $u$  the number of its nodal domains is at least*

$$\max\{2 \operatorname{ord}_x(u) + 2\bar{\chi} - 2l - 2 \mid x \in \mathcal{N}^2(u)\},$$

where  $\bar{\chi}$  is the reduced Euler-Poincaré number of  $M$  and  $l$  is the number of the boundary components.

We proceed with demonstrating the second inequality in Theorem 1.3.1:

$$m_k(g, \mu) \leq 2(2 - \bar{\chi}) + 2l + k.$$

Suppose the contrary to its statement. Then there exists at least  $2(2 - \bar{\chi}) + 2l + k + 1$  linearly independent eigenfunctions corresponding to the eigenvalue  $\sigma_k(g, \mu)$ . Pick an interior point  $x \in M$ . By Lemma 3.2.2 there exists a new eigenfunction  $u$  whose vanishing order at the point  $x$  is at least  $2 - \bar{\chi} + l + [(k + 1)/2]$ , where the brackets stand for the integer part. Using the estimate in Lemma 3.3.1, we see that the number of the nodal domains of  $u$  is at least  $k + 2$ . Thus, we arrive at a contradiction with Courant's nodal domains theorem.  $\square$

*Remark 3.3.2.* It is clear from the proof that actually a slightly improved inequality holds for the Steklov eigenvalue multiplicities:

$$m_k(g, \mu) \leq 2(2 - \bar{\chi}) + 2l + k - [k]_2,$$

where  $[n]_2$  is a function that takes values 0 or 1 with respect to the cases  $n$  is odd or even respectively.

The rest of the section is devoted to the proof of Lemma 3.3.1. We start with the following lemma, which is specific to the Steklov eigenvalue problem.

**Lemma 3.3.3.** *Let  $u$  be a non-trivial Steklov eigenfunction and  $x \in \mathcal{N}^2(u)$  be a vertex in its nodal graph. Further, let  $\Gamma_1$  be a subgraph of  $\mathcal{N}(u)$  that is the union of all circuits in the connected component of  $x$  and all simple paths joining  $x$  and the vertices of these circuits. Then the degree of  $x$  in  $\Gamma_1$  is at most  $2l + 2 - 2\tilde{\chi}$ .*

*Proof.* Let  $v_1$ ,  $e_1$ , and  $f_1$  be the number of vertices, edges, and faces of  $\Gamma_1$  respectively. Since every vertex in  $\Gamma_1$ , different from  $x$ , belongs either to a circuit or the interior of a simple path, its degree in  $\Gamma_1$  is at least 2. Thus, by formula (2.3.1) we have

$$(3.3.4) \quad 2e_1 \geq \deg_{\Gamma_1}(x) + 2(v_1 - 1).$$

Recall that every nodal domain of  $u$  has a non-trivial arc on the boundary. Since a face of  $\Gamma_1$  contains the union of nodal domains, it contains at least one boundary component of  $M$ . Since two faces can not contain the same boundary component, we have  $f_1 \leq l$ . Viewing  $\Gamma_1$  as a subgraph in the reduced nodal graph  $\mathcal{N}(u)$ , we can apply the Euler inequality to obtain

$$e_1 \leq v_1 + f_1 - \tilde{\chi} \leq v_1 + l - \tilde{\chi}.$$

Now the statement follows by the combination of this inequality with relation (3.3.4).  $\square$

*Proof of Lemma 3.3.1.* Let  $x \in \mathcal{N}^2(u)$  be a vertex in the nodal graph, and  $\Gamma_1$  be a subgraph from Lemma 3.3.3. Further, let  $\Gamma_2$  be a subgraph formed by all simple paths in the nodal set starting from  $x$  and approaching the boundary  $\partial M$  that do not intersect  $\Gamma_1$  except for  $x$ . This subgraph does not contain any circuits, and hence, is a tree. We denote by  $v_2$  and  $e_2$  the number of its vertices and edges respectively. Clearly, any nodal edge incident to  $x$  belongs either to  $\Gamma_1$  or  $\Gamma_2$ , that is

$$(3.3.5) \quad 2 \operatorname{ord}_x(u) = \deg_{\Gamma_1}(x) + \deg_{\Gamma_2}(x).$$

We claim that the number of edges in  $\Gamma_2$  that are not incident to  $x$  is at least one less than the number of vertices,

$$(3.3.6) \quad e_2 - \deg_{\Gamma_2}(x) \geq v_2 - 1.$$

Indeed, this follows from the fact that  $\Gamma_2$  is a tree, and that edges approaching the boundary have only one vertex.

Finally, consider a subgraph  $\Gamma$  that is the union of  $\Gamma_1$  and  $\Gamma_2$ . We use the notation  $v$ ,  $e$ , and  $f$  for the number of its vertices, edges, and faces respectively. Clearly, we have

$$e = e_1 + e_2, \quad v = v_1 + v_2 - 1.$$

Combining these identities with relations (3.3.4)-(3.3.6), we obtain

$$e - v \geq 2 \operatorname{ord}_x(u) - \frac{1}{2} \deg_{\Gamma_1}(x) - 1.$$

Using the bound for the degree from Lemma 3.3.3, we arrive at the relation

$$e - v \geq 2 \operatorname{ord}_x(u) + \bar{\chi} - l - 2.$$

Finally, viewing  $\Gamma$  as a subgraph in the reduced nodal graph  $\mathcal{N}(u)$ , we combine the last relation with the Euler inequality to obtain

$$f \geq e - v + \bar{\chi} - l \geq 2 \operatorname{ord}_x(u) + 2\bar{\chi} - 2l - 2.$$

Since the number of faces  $f$  is not greater than the number of nodal domains, we are done.  $\square$

#### 4. ASYMPTOTIC MULTIPLICITY BOUNDS

**4.1. Proof of Theorem 1.4.1.** Let  $D$  be a “weighted” Dirichlet-to-Neumann operator on  $\partial M$  defined as

$$C^\infty(\partial M) \ni u \mapsto Du = \rho^{-1} \frac{\partial \hat{u}}{\partial \nu} \in C^\infty(\partial M),$$

where  $\hat{u}$  denotes the unique harmonic extension of  $u$  into  $M$ . When  $\rho$  is smooth and positive, it defines a self-adjoint elliptic pseudo-differential operator of the first order whose eigenvalues coincide with the Steklov eigenvalues, see [26, pp. 37-38] and [24]. Let  $N(\lambda)$  be its eigenvalue counting function; it equals the number of eigenvalues counted with multiplicity that is strictly less than a positive  $\lambda$ . By the result of Hörmander in [19, 25], the function  $N(\lambda)$  satisfies the following asymptotics (Weyl law):

$$(4.1.1) \quad N(\lambda) = \frac{\lambda}{2\pi} \int_{\partial M} \rho(s) ds_g + R(\lambda),$$

where  $R(\lambda)$  is a bounded quantity in  $\lambda > 0$ . Using this formula, we obtain

$$\begin{aligned} m_k(g, \mu) &= \lim_{\varepsilon \rightarrow 0} N(\lambda_k + \varepsilon) - N(\lambda_k) = \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2\pi} \int_{\partial M} \rho(s) ds_g + R(\lambda_k + \varepsilon) - R(\lambda_k) \leq 2 \sup |R(\lambda)|. \end{aligned}$$

Thus, the multiplicity  $m_k(g, \mu)$  is indeed bounded, and the theorem is demonstrated.  $\square$

It is interesting to know up to what extent the bound on  $m_k(g, \mu)$  depends on a metric and a boundary measure; in particular, whether there exists a universal constant (possibly depending on the genus of  $M$ ) for which Theorem 1.4.1 holds.

**4.2. Proof of Proposition 1.4.2.** By the uniformisation theorem, we may assume that  $M$  is the unit disk and the metric  $g$  on  $M$  is conformal to the Euclidean metric  $g_{Euc}$ . Since the Dirichlet energy is conformally invariant, from the variational principle it follows that the Steklov eigenvalues of  $(M, g)$  with a weight function  $\rho$  coincide with the Steklov eigenvalues of  $(M, g_{Euc})$  with the a new weight function  $\rho_0$  that depends on  $\rho$  and the values of  $g$  on  $\partial M$  only. By the results in [24, 8], the latter satisfy the following refinement of Weyl’s asymptotic formula:

$$(4.2.1) \quad \sigma_{2k} = \frac{2\pi k}{\int_{\partial M} \rho_0(s) ds} + o(k^{-\infty}), \quad \sigma_{2k+1} = \frac{2\pi k}{\int_{\partial M} \rho_0(s) ds} + o(k^{-\infty}),$$

as  $k \rightarrow \infty$ . Thus, we conclude that for a large  $k$ , the eigenvalues  $\sigma_k$  may be at most double. Note that for a disk all nonzero eigenvalues have multiplicity two, and therefore this result is sharp.  $\square$

*Remark 4.2.2.* The hypotheses of Proposition 1.4.2 on the smoothness of  $\partial M$  and  $\rho > 0$  are essential for the asymptotic formula (4.2.1) to hold. Even for domains with piecewise smooth boundaries the asymptotic properties of the spectrum may be quite different. In particular, by a direct computation one can show that formulas (4.2.1) fail for a square: for a large  $k$  the Steklov spectrum of a square is a union of quadruples of eigenvalues such that in each quadruple the eigenvalues are  $o(k^{-\infty})$ -close [13]. However, no counterexample to Proposition 1.4.2 is known for simply-connected surfaces with non-smooth boundaries, and it would be interesting to understand whether the result holds in this case as well.

## 5. OTHER BOUNDARY VALUE PROBLEMS

**5.1. Eigenvalue problems with homogeneous boundary conditions.** The method used to prove the first inequality in Theorem 1.3.1 uses only Courant's nodal domain theorem and the behaviour of eigenfunctions in the interior of  $M$ ; it largely disregards their behaviour on the boundary. The purpose of this section is to show that it applies to rather general boundary value problems.

Let  $(M, g)$  be a compact Riemannian surface with boundary and  $L = (-\Delta_g) + V$  be a Schrodinger operator, where  $V$  is a smooth potential. Denote by  $B$  a boundary differential operator of the form

$$Bu = au + b \frac{\partial u}{\partial \nu},$$

where  $a$  and  $b$  are smooth functions on  $\partial M$  and at least one of them does not have zeroes. We consider the following eigenvalue problem

$$(5.1.1) \quad Lu = \lambda u \quad \text{in } M, \quad \text{and} \quad Bu = 0 \quad \text{on } \partial M;$$

it is often referred to as the Robin boundary value problem, and in particular, contains the Dirichlet and Neumann problems as its special cases. By

$$0 \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k \leq \dots$$

we denote the corresponding eigenvalues.

The following statement gives a bound for the eigenvalue multiplicities of problem (5.1.1) that is independent of a Schrodinger operator  $L$  and, more interestingly, of a boundary operator  $B$ .

**Proposition 5.1.2.** *Let  $M$  be a smooth compact surface with a non-empty boundary. Then for any Schrodinger operator  $L$  and any Robin boundary operator  $B$  the multiplicity  $m_k$  of an eigenvalue  $\lambda_k$  corresponding to problem (5.1.1) satisfies the inequality*

$$(5.1.3) \quad m_k \leq 2(2 - \bar{\chi}) + 2k + 1,$$

where  $\bar{\chi} = \chi + l$ , and  $\chi$  and  $l$  stand for the Euler-Poincare number and the number of the boundary components of  $M$  respectively.

**5.2. Details on the proof.** We explain how the arguments and results in sections 2 and 3 carry over this general eigenvalue problem. First, Proposition 2.2.1 holds for solutions of second order elliptic differential equations with smooth coefficients. In particular, it holds for eigenfunctions of problem (5.1.1). Thus, the nodal set of an eigenfunction has a similar graph structure. These eigenfunctions also enjoy Courant's nodal domain theorem, see [7], and the arguments in section 3 show that their nodal graphs are finite. The version of Proposition 2.2.1 also shows that the statement of Lemma 3.2.2 holds for solutions of general second order elliptic equations, cf. [23, Lemma 4]. The rest of the proof of the first inequality in Theorem 1.3.1 carries over without changes.

Finally, mention that inequality (5.1.3) holds also for eigenvalue problems with mixed boundary conditions. In addition, one can also allow non-smooth boundaries as long as the eigenvalue problem remains well-posed and Courant's nodal domain theorem holds.

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DEPARTMENT OF GEOMETRY AND TOPOLOGY, MOSCOW STATE UNIVERSITY, LENINSKIE  
GORY, GSP-1, 119991, MOSCOW, RUSSIA  
*E-mail address:* karpukhin@mccme.ru

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, D-80333  
MÜNCHEN, GERMANY  
*E-mail address:* Gerasim.Kokarev@mathematik.uni-muenchen.de

DÉPARTEMENT DE MATHÉMATIQUES ET STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, CP 6128  
SUCC CENTRE-VILLE, MONTRÉAL, QC H3C 3J7, CANADA  
*E-mail address:* iossif@dms.umontreal.ca